How Long is One Inch?

Someone who measures the length of an object typically does so by holding the object up to a ruler and reading the length from the ruler. If the length of the object matches the one inch length on the ruler, then, the object is one inch long. What can be more obvious than this?

Saying exactly how long one inch is, however, may not be as easy as it seems. The problem with specifying the exact length of one inch has to do with the nature of boundaries. If someone holds up a ruler to a sheet of paper, where is the correct place to begin measuring? Does a one inch long object span the distance between the inner boundaries of the one inch markers? Or, is it more correct to measure from the middle of one inch marker to the middle of the other? Or, instead, is either correct, provided that the beginning lies somewhere on one boundary and the ending lies somewhere on the other boundary?

Why does specifying exactly where to begin and end measuring not seem to be a problem? The reason is that we often treat the boundary lines on the ruler as defining an imaginary line that has no width. Thus, the border area between one inch and two inches on a ruler is conceived as having no width, so that the question of where to begin and end measuring simply does not arise.

The tradition of treating a boundary line as having no width is as old as Euclid's definition of a line as "a breadthless length." If a line is "breadthless," i.e. has no width, and a boundary is a line, then surely it is correct to treat the boundary as a line with no width.

Treating a boundary line as one with no width works quite well in some cases. It works well in cases when the boundary line is so thin relative to what it is a boundary of that no purpose would be served by treating the boundary line as having width. For example, a piece of rope that separates two tracts of land may be so thin relative to the size of the land that no purpose would be served by trying to specify the boundary more precisely. Even if there is a small portion of land that lies directly on this boundary, this portion is so small that it can be treated as nonexistent for the purposes of dividing the two tracts of land.

The case may be different when what lies on the boundary becomes important, or when the boundary line is large relative to the size of the area whose boundary is marked. If gold lies on the boundary line between two properties, it may become important to try to specify for each portion of the gold whose land it lies on. Another example is the center line on the highway. This line is significant in size relative to the width of the road. The center line is a dividing line between the two sides of highway that doesn't belong to either side. This center line may only be several inches wide, but it is an example of a boundary line with width. A doorway between two rooms provides a similar example of a boundary that has width.

Does it ever matter exactly where we should begin measuring the length of one inch, one foot, one meter, or any other length? Often it doesn't matter, so where's the problem? One problem is that mathematics is supposed to be a model of certainty and precision, so that showing that there is an element of uncertainty and imprecision in our most fundamental methods of measuring is disquieting. If we are unable to specify the exact length of our most fundamental unit of measurement, this introduces an element of inexactness in all measurements that depend on that unit.

Since the idea that there is no problem with the boundaries of measurement depends on the belief that a boundary marks an imaginary line with no width, let us examine this belief. Children are taught in geometry class that mathematics is about imaginary and ideal objects, not about real, physical objects. A square object is a square if it closely approximates the dimensions of the "ideal square," even if it is not exactly square. It is only natural, then, to extend this idea to boundary lines, and to treat boundaries as lines with no width. Lines have only one dimension, length, and they mark an invisible "ideal line" that has no width, even if the physical lines themselves have a narrow width.

How does this work in practice? When we divide a circle into two equal parts by drawing a diameter from one side to another, what about the area that lies on the dividing line? Which side of the circle does this area belong to? For that matter, does the diameter end at the beginning of the circular line, the middle of it, or the end? Does the area of a circle include the area of the line that defines the circle, or is the area simply the area inside the circle? None of these questions arise if we treat a line as having no width. Yet, they all arise if we recognize a line as having width.

Does it make sense to treat lines as having width? To answer this question, let us look at the function of measurement. To measure an object is typically to find out how many units it contains, where the unit is some unit of length, volume, or other unit of measurement. When someone is baking a cake, for example, that person wants to know how many cups of flour to put into the cake. Likewise, quantities are important in commerce. A customer who buys a gallon of milk wants to know that he is getting one gallon, not some percentage of a gallon. One function of measurement, then, is to specify quantities for practical matters such as recipes, and to insure that people get the advertised quantities of products.

If we treat lines as having no width, this may have no practical impact in some situations. If someone wishes to divide a piece of cake into two equal slices, he or she may simply mark a line in the middle and physically divide the cake by cutting along the line. This act of division forces all particles into one side or the other, and creates two pieces of cake where formerly there was one. Of course, some "crumbs" may result which are the particles of cake that don't stick to one piece or the other; these are the byproducts of the division process.

When the quantities are not being physically divided but only divided by a line, as in the border between two towns, the width of the line may make a difference. In some cases, where the border is disputed, a no-man's-land may be specified to mark an area between two provinces or countries that belongs to neither one. And in mathematical examples, theoretical problems arise in specifying the exact border or boundary of geometrical objects. It is reasonable to wonder, for example, whether the area of a circle includes only the area within the circle, or whether it includes the border of the circle as well. This is especially true since the area of a circle is, by conventional mathematics, specified by an irrational number, so that clarifying exactly what the reference of the expression "area of a circle" is might shed some light on our inability to specify this area with rational numbers.

Whether a line has width depends on what me mean by "line." Adolf Grunbaum comments on this issue. Speaking in context of a discussion of Cantor's set theory, he says:

> No clear meaning can be assigned to the "division" of a line unless we specify whether we understand by "line" an entity like a sensed "continuous" chalk mark on the blackboard or the very differently continuous line of Cantor's theory. The "continuity" of the sensed linear expanse consists essentially in its failure to exhibit visually noticeable gaps as the eye scans it from one of its extremities to the other. There are no distinct elements in the sensed "continuum" of which the seen line can be said to be a structured aggregate.

Does the idea that a line has no width make sense? Of course, it is possible to treat a line as having no width for the purposes of measurement. But it is not possible to draw a line with no width. A line with no width is no line at all. Any line, no matter how thin, has some width. The width of a line is parallel to the duration of a unit of time. It is not possible to specify any period of time that does not have duration. One hour, one minute, one second, one millisecond, and one nanosecond all have some duration. It is simply not possible to specify a unit of time that has no duration. Likewise, it is not possible to specify a line that has no width, however small that width may be.

Doesn't saying that a line must have width depend on confusing the width of an actual physical line with the width of the ideal line it stands for? While treating a dividing line or boundary as standing for an invisible "ideal line" may work satisfactorily in some situations, it isn't always satisfactory. It isn't satisfactory in some cases because when it is physically impossible to separate two quantities, treating a line as marking an ideal though invisible line between two quantities doesn't say which quantity the matter that lies <u>on</u> the dividing line belongs to. No matter how thin the line is, there will always be some material that lies on the line, just as the diameter or the edge of a circle occupies some area. The idea that a line has no width is a mathematical fiction that simply ignores the fact that boundaries by their very nature take up space.

How did the idea that boundaries have only one dimension become so deeply ingrained in our thinking? The answer lies in the way we mathematically conceive of points and lines. In mathematics, a point is conceived of as having only the attribute of location; it does not take up space. Likewise, a line is conceived of as made up of infinitely many of these no-dimensional points. If a point has no area or width, then surely a line that is made up of these no-dimensional points will have no width. Yet, this conception ignores a basic mathematical fact: a large number, or even an infinite number, of points that have no area do not acquire the property of area. Multiplying zero infinitely many times by itself still equals zero. Hence, it is a mistake to conceive of a line as <u>being made up of</u> infinitely many points; instead, a line should be conceived of as a quantity that represents the distance between points. The definition of a circle as "a set of points equidistant from a fixed point" ignores this mathematical fact. If points have no area, then infinitely many of them don't have area either.

Adolf Grunbaum addresses the question whether a line that is made up of unextended points can be said to have a length. According to Grunbaum, it is perfectly consistent to maintain that a line is made up of unextended points and that the line has length. Grunbaum introduces the subject as follows:

> It is a commonplace in the analytic geometry of physical space-time that an extended straight line segment, having positive length, is treated as "consisting of" unextended points, each of which has zero length. Analogously, time intervals of positive duration are resolved into instants, each of which has zero duration.

Grunbaum invokes Zeno in explaining the difficulty in reconciling line segments made up of unextended points with the idea that these line segments have length:

> Zeno invokes two basic axioms in his mathematical paradoxes of plurality. Having divided all magnitudes into positive and "dimensionless" magnitudes, Zeno assumed that:
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> 1) The sum of an infinite number of equal positive magnitudes of arbitrary smallness must necessarily be infinite

2) The sum of any finite or infinite number of "dimensionless" magnitudes must necessarily be zero ... Zeno himself is presumed to have used these axioms as a basis for the following dilemma: If a line segment is resolved into an aggregate of infinitely many <u>like</u> elements, then two and only two cases are possible. Either these elements are of equal positive length and the aggregate of them is of infinite length (by Axiom 1) or the elements are of zero length and then their aggregate is necessarily of zero length (by Axiom 2).

Treating a point as having no area also generates mathematical paradoxes, such as Zeno's paradox. The idea that motion is impossible because between any two points it is possible to insert another point is plausible only if one treats location as being specifiable by a mathematical point. If my location is specifiable by a mathematical point with no area, then, since it is always possible to insert a mathematical point between two other mathematical points, it is possible to generate an argument that it is impossible for me to move from point A to point B. Once it is recognized that when someone's location is specified, some unit of measurement must be given, and this unit of measurement specifies some portion of space, however small, the paradoxical nature of the argument disappears. It disappears because there will always be a finite number of units of length between two locations, whether these units are inches, feet, meters, miles, or any other unit of measurement. If two objects are six inches apart, one might argue that the first object must move three inches, then four and one half inches, then five and one quarter inches, etc. But the argument eventually breaks down because, at some point, moving another mathematically possible distance won't count as a move at all because the move will be too small to be captured by the selected unit of measurement. Moving from 5 999/1000 to 5 9995/10000 won't count as a move if the unit of measurement is in inches, since both of these positions will be so close to six inches that they will count as being at six inches. Zeno's paradox is generated by changing the units of measurement each time to make them more precise to infinity. Zeno's argument doesn't work if the units of measurement are specified ahead of time.

The problem is parallel to adding one drop of milk to one gallon of milk. If the unit of measurement is in gallons, adding or taking away a drop of milk won't make the measurement greater than or less than one gallon, even though the actual physical quantity is different. The reason is that the unit of measurement isn't precise enough to capture the difference the addition of one drop to a gallon of milk makes. If one is measuring milk in drops, this unit would be precise enough to capture the difference. It would not be precise enough, however, to capture the addition of 1/1000th of a drop.

How does this reasoning apply to the question "How long is one inch?" In order to specify precisely the length that one inch takes up, we need to specify the exact width of the boundary. Once it is recognized that boundaries have width, and this width is specified, then the difference between two objects that differ in length by a quantity that is less than the specified width of the boundary is parallel to the difference between two gallons of milk, one of which has one more drop than the other. The difference will be too small to be captured by the precision of the chosen unit of measurement. In order to capture this difference, it is necessary to introduce a more fine-grained unit of measurement. Likewise, if I say "It took me fifteen minutes to take out the laundry," my statement wouldn't be falsified if it had taken me fifteen minutes and one second. As long as the unit of measurement is in minutes, 15 minutes, one second, and 15 minutes, two seconds, will both count as fifteen minutes. To capture the difference between the two cases, it would be necessary to use seconds, not minutes, as the unit of measurement.

How long, then, is one inch? Once the width of the boundary line demarcating inches is specified, then any length whose border lies on the boundary line will equal one inch. This means that two objects can have different lengths, yet both be one inch long. In order to capture the difference in lengths, it would be necessary to introduce a narrower boundary line.

The question about the area of a circle seems more unclear. Possibly, we should distinguish "the area within a circle," which does not include the boundary line, from "the area of a circle," which does include it. The diameter of a circle, then, would end anywhere on the boundary of the circle.

What would be the implications for geometry of accepting the idea that lines have width? It would mean giving up the fiction that it is possible to draw a line dividing two areas that is infinitely small, or that has no width. It may also mean that we may need to develop a new geometry of boundaries and borders, based on the idea that boundaries and borders have width, however small that width may be.

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